COHEN-MACAULAY RINGS SELECTED EXERCISES

KELLER VANDEBOGERT

1. Problem 1.1.9

Proceed by induction, and suppose $x \in R$ is a U and N-regular element for the base case. Suppose now that $xm = 0$ for some $m \in M$. We want to show that $m = 0$. Then, let $\varphi : U \to M$ and $\psi : M \to N$ be the sequence maps. Note that by properties of exact sequences that ψ is surjective and φ is injective.

Since $xm = 0$, $\psi(xm) = x\psi(m) = 0 \implies \psi(m) = 0$, since x is regular in N. Thus, $m \in \operatorname{Ker}\psi = \varphi(U)$ so that $m = \varphi(u)$ for some $u \in U$.

Then, certainly $xm = \varphi(xu) = 0$, and by injectivity, we know that $xu = 0$. Since x is regular, we conclude that $u = 0$ so that $m = \varphi(0) = 0$, and x is regular in M.

Suppose now that $\mathbf{x} = x_1, \dots, x_n$ is weak U and N regular for all $i < n$. Then, merely apply the argument of the base case to the exact sequence induced:

$$0 \to U/(x_1, \dots, x_{n-1})U \to M/(x_1, \dots, x_{n-1})M \to N/(x_1, \dots, x_{n-1})N \to 0$$

To show that $\mathbf{x}$ must be a weak M regular sequence as well.

Now, suppose that $\mathbf{x}$ is weak U-regular and N-regular. Using Proposition 1.1.4, we have an induced exact sequence

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$$0 \to U/\mathbf{x}U \to M/\mathbf{x}M \to N/\mathbf{x}N \to 0$$

Suppose for contradiction that $M/\mathbf{x}M = 0$. Then, by properties of exactness, the induced map $\bar{\psi}$ is surjective and we conclude $N/\mathbf{x}N = 0$, contradicting the fact that $\mathbf{x}$ is N-regular. Thus, $\mathbf{x}$ is also M-regular.

2. Problem 1.1.10

(a). Suppose that x and x' are both M-regular elements. Suppose $(xx')m = 0$ for some $m \in M$. Then, since x is regular, $x'm = 0$, and since x' is regular, we conclude $m = 0$. Thus, xx' is M-regular, and the general case for weak M-regular sequences follows by induction.

Now, assume that $M/xM \neq 0$ and $M/x'M \neq 0$, and suppose for contradiction that $M/xx'M = 0$. Then, since $M = xx'M$ and $x'M \subset M$, we multiply by x to see that $M \subset xM$ so that $M/xM = 0$ (and by commutativity we also see $M/x'M = 0$), a clear contradiction. Thus, $M/xx'M \neq 0$ so that the general case follows by induction.

(b). This result follows from part (a) almost immediately, where we note that we can apply this to any M-regular sequence and itself, merely identifying the ith element each time. More precisely, suppose $\mathbf{x} = x_1, \dots, x_n$ is regular. Then, by induction and the result of (a), $x_1^{e_1}, x_2, \dots, x_n$ is regular as well, with e_1 some integer $\geqslant 1$. Now successively apply this for every other index $1 < i \leqslant n$ to see that $x_1^{e_1}, x_2^{e_2}, \dots, x_{n-1}^{e_{n-1}}, x_n^{e_n}$ is also M-regular.

3. Problem 1.1.11

We want to prove the following: Suppose $\mathbf{x}$ is a weak-$M \otimes_R N$ sequence, and N is faithfully flat over R. Then $\mathbf{x}$ is a weak M-sequence.

Let x be a regular $M \otimes N$ element (as usual, the general case follows from induction). We have the homothety $\psi : M \otimes N \to M \otimes N$, $m \otimes n \mapsto x(m \otimes n)$, which is injective since x is regular. By definition, $x(m \otimes n) = (xm) \otimes n$ so that $\psi(m \otimes n) = (xm) \otimes n$.

By faithful flatness of N, the map $x \otimes \mathrm{id}_N$ is injective iff x is injective ($x : M \to M$, $m \mapsto xm$). But this is equivalent to saying that the element x is M-regular, so we are done.

4. Exercise 1.4.18

Since M has a rank, say r, we have that $M \otimes Q = Q^r$, Q denotes our field of fractions of R. We can choose a basis $\{e_1, \ldots, e_r\}$ for the above vector space. Since M is finitely generated, we can also choose a generating set $\{x_1, \ldots, x_n\}$. Consider the inclusion $M \hookrightarrow Q^r$. We can find $a_{ij}, b_{ij} \in R$ such that $x_i = \sum_j \frac{a_{ij}}{b_{ij}} e_j$. Take $b := \prod_{i,j} b_{ij}$. We can then consider the inclusion

$$M \hookrightarrow \frac{Re_1}{b} \oplus \frac{Re_2}{b} \oplus \cdots \oplus \frac{Re_r}{b} \cong R^r$$

The above is obviously a free module, and the inclusion is injective since M is torsion free. Therefore, M is isomorphic to its image, which is a submodule of the free module R^r. Note also that R^r clearly has the same rank, as $R^r \otimes Q = Q^r$, so the problem is solved.

5. Problem 1.4.19

Since R is Noetherian and M, N are finitely generated, M is finitely presented. Choose a presentation

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

Apply $\operatorname{Hom}_R(-, N)$ to the above, and recall that $\operatorname{Hom}_R(R, N) \cong N$. We will have an induced short exact sequence (with N' being the image of the map from $N^n \to N^m$):

$$0 \longrightarrow \operatorname{Hom}(M, N) \longrightarrow N^n \longrightarrow N' \longrightarrow 0$$

Employing the result of Proposition 1.2.9 of the book, we see

$$\operatorname{grade}(I, \operatorname{Hom}(M, N)) \geqslant \min\{\operatorname{grade}(I, N),\ \operatorname{grade}(I, N') + 1\}$$

Now we have 3 cases: in the trivial case $\operatorname{grade}(I, N) = 0$, obviously $\operatorname{grade}(I, \operatorname{Hom}(M, N)) \geqslant 0$ always. Suppose $\operatorname{grade}(I, N) = 1$. Then $\operatorname{grade}(I, N') = 1$ as well, and the result again follows immediately. Finally, when $\operatorname{grade}(I, N) \geqslant 2$, we have that $\operatorname{grade}(I, N') \geqslant 1$, so that

$$\operatorname{grade}(I, \operatorname{Hom}(M, N)) \geqslant \min\{\operatorname{grade}(I, N),\ 2\}$$

Completing the proof.

6. Problem 1.4.20

(a). First, we have the following: *Claim: If M is finitely generated and R is Noetherian, then M^* is torsion free and finitely generated.*

To prove this claim, note that since M is finitely generated, we have an exact sequence

$$R^n \longrightarrow M \longrightarrow 0$$

Applying $\operatorname{Hom}_R(-, R)$ to the above yields

$$0 \longrightarrow M^* \longrightarrow R^n$$

Hence, M is isomorphic to its image in a torsion free submodule. Since R is Noetherian, every submodule of R^n is finitely generated, giving that M^* is torsion free and finitely generated.

Now we can proceed to the problem. Assume M is torsionless and suppose that $rm = 0$ for $0 \neq r \in R$ and $m \in M$. We want to prove that $m = 0$. Consider the map $m \mapsto (\phi(m))_{\phi \in M^*}$. By definition of torsionless, the image of m is 0 iff $m = 0$ identically. Consider $rm \mapsto (\phi(rm))_{\phi \in M^*} = r(\phi(m))_{\phi \in M^*}$. Since $rm = 0$, this point maps to 0. However, the above shows that $r(\phi(m))_{\phi \in M^*} = 0$. Since M^* is torsion free, we conclude that $m = 0$, so that M is torsion free as asserted.

(b). First note that if M is a submodule of a finite free module R^n, then it is obviously torsionless since $(R^n)^{**} \cong R^n$, so that any free module is torsionless (and hence any submodule of a free module. *Note that this improves the claim of part* (a), *as we now see that* M^* *is also torsionless.*

Now, suppose that M is torsionless. By the claim proven in part (a), M^* is finitely generated and we can choose a generating set $\{\phi_1, \ldots, \phi_n\}$. Then, we can consider the following map $\Theta : M \to R^n$, which takes $\Theta(m) = (\phi_i(m))_{i=1}^n$. It suffices to show that this map is injective, so that M is isomorphic to its image as a submodule of a free module.

Suppose then that $\Theta(m) = 0$. Then, $\phi_i(m) = 0$ for each generator. But then we see that $\phi(m) = 0$ for all $\phi \in M^*$. Since M is torsionless, $m = 0$, so that Θ is injective, as desired.

(c). Note that since M^* is finitely generated, the claim of part (a) along with the remark in (b) gives that M^{**} is finitely generated and torsionless. Since M is reflexive, $M = M^{**}$, so M is torsionless and hence isomorphic to a submodule of a free module (by (b)). Taking F_0

as the cokernel of the map $M \hookrightarrow R^n := F_1$, we have the exact sequence

$$0 \longrightarrow M \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

So that M is a second syzygy.

7. Problem 1.4.21

We have an exact sequence

$$F \longrightarrow G \longrightarrow M \longrightarrow 0$$

Dualizing yields

$$0 \longrightarrow M^* \longrightarrow G^* \longrightarrow F^* \longrightarrow D(M) \longrightarrow 0$$

We have that M^* is finitely presented, so choose a presentation $F_1 \longrightarrow F_0 \longrightarrow M^* \longrightarrow 0$, leading to a partial free resolution of $D(M)$:

$$F_1 \longrightarrow F_0 \longrightarrow G^* \longrightarrow F^* \longrightarrow D(M) \longrightarrow 0$$

Dualizing our presentation of M^*, we get the exact sequence

$$0 \longrightarrow M^{**} \longrightarrow F_0^* \longrightarrow F_1^*$$

Leading to the sequence

$$G \longrightarrow M \xrightarrow{h} M^{**} \longrightarrow F_0^*$$

Which can be extended to the sequence

$$F \longrightarrow G \longrightarrow F_0^* \longrightarrow F_1^*$$

Now, let us consider computing the cohomology in the above sequence. The cohomology at G is going to be the kernel of the map taking $g \mapsto g + \operatorname{Im}\phi \mapsto h(g + \operatorname{Im}\phi) \hookrightarrow F_0^*$ modulothe image of ϕ, which is M. That is, $g + \operatorname{Im}\phi \in \operatorname{Ker} h$, so that $\operatorname{Ker}(G \to F_0^*) = \{g : g + \operatorname{Im}\phi \in \operatorname{Ker} h\}$. Hence, taking the quotient $\operatorname{Ker}(G \to F_0^*)/\operatorname{Im}\phi$ gives precisely $\operatorname{Ker} h$.

Now consider the cohomology at F_0^*. $\mathrm{Im}(G \to F_0^*)$ is isomorphic to $\mathrm{Im}\, h$ viewed as a submodule of F_0^* (under the inclusion $M^{**} \hookrightarrow F_0^*$). Also, by the exact sequence $0 \longrightarrow M^{**} \longrightarrow F_0^* \longrightarrow F_1^*$, we have that $\mathrm{Ker}(F_0^* \to F_1^*) = M^{**}$.

Now, to solve the problem, dualize our partial resolution for $D(M)$. We then see that the above cohomology groups are precisely isomorphic to $\mathrm{Ext}^1_R(D(M), R)$ and $\mathrm{Ext}^2_R(D(M), R)$, respectively.

8. Exercise 1.4.22

9. Exercise 1.4.23

10. Exercise 1.4.24

11. Problem 3.1.21

Since R is given as a PID, it suffices to show the quotient K/R of the field of fractions by R is divisible. Note:

$$K/R = \{r/s \in K/R \mid s \neq 1\}$$

Hence, given $r/s \in K/R$, we can write this as $r(1/s)$, $r \in R$, hence every element is divisible so that K/R is injective (and hence $0 \longrightarrow K \longrightarrow K/R \longrightarrow 0$ is an injective resolution).

12. Problem 3.1.22

We proceed by proving that $\mathrm{Hom}_R(R, k)$ is the injective hull of the residue field, which is indecomposable by Theorem 3.2.6.

Take $S := \mathrm{Soc}(\mathrm{Hom}_R(R, k))$. First note that injectivity is clear, since

$$\begin{aligned}\mathrm{Hom}_R(-,\mathrm{Hom}_k(R,k)) &\cong \mathrm{Hom}_R(- \otimes_k R, k)\\ &\cong \mathrm{Hom}_R(-,k)\end{aligned}$$

Since k is a field, it is obvious that $\mathrm{Hom}_R(-,k)$ is exact, so that $\mathrm{Hom}_R(R,k)$ is an injective module over R. Continuing, since R has finite k-dimension, $\mathrm{Hom}_k(R,k)$ is an Artinian module, so that S is the intersection of all essential extensions of $\mathrm{Hom}_R(R,k)$, however is also characterized as

$$\begin{aligned}S = \mathrm{Hom}_R(R/\mathfrak{m},\mathrm{Hom}_k(R,k) &\cong \mathrm{Hom}_R(k \otimes_k R, k)\\ &\cong \mathrm{Hom}_k(k,k) \cong k\end{aligned}$$

From the set of all simple submodules of $\mathrm{Hom}_k(R,k)$, we can choose a minimal element M. By definition of essential extension, $M \cap S \neq 0$ and hence we see that $\mathrm{Hom}_k(R,k)$ is an injective essential extension of S, so that $\mathrm{Hom}_k(R,k) = E_R(S) = E_R(k)$, which is nondecomposable, so we are done.

13. Problem 3.1.23

Suppose there exists a nonzero finitely generated injective module E. We have that $\dim_R(E) \leqslant \mathrm{id}_R(E) = 0$, so that E is an Artinian module (since E is also finitely generated) with $\mathrm{Supp}(E) = \{\mathfrak{m}\}$. By 3.2.8, $E \cong \oplus E(R/\mathfrak{m})$, implying that $E(R/\mathfrak{m})$ must also be finitely generated.

If $T(E)$ denotes our Matlis dual (so that $T(-) = \mathrm{Hom}(-, E(k))$), we see that $T(E(k))$ must be Artinian as an R-module. However, $T(R) \cong E(k)$, and hence:

$T(E(k)) = T(T(R)) \cong R$ So that R is Artinian as an R-module, and hence as a ring.

14. Problem 3.1.24

We proceed by induction on $\operatorname{depth}_R M$. Assume $M \neq 0$ (else the statement holds trivially) and suppose first that $\operatorname{depth}_R M = 0$. Then, $\mathfrak{m} \in \operatorname{Ass}(M)$ so that we have an exact sequence

$$0 \longrightarrow R/\mathfrak{m} \xrightarrow{x} M \longrightarrow L \longrightarrow 0$$

Set $r := \operatorname{id}_R N$, and apply $\operatorname{Hom}_R(-, N)$ to the above. We have an induced exact sequence:

$$\cdots \longrightarrow \operatorname{Ext}^r_R(M, N) \longrightarrow \operatorname{Ext}^r_R(R/\mathfrak{m}, N) \longrightarrow \operatorname{Ext}^{r+1}_R(L, N) \longrightarrow \cdots$$

Obviously $\operatorname{Ext}^r_R(R/\mathfrak{m}, N) \neq 0$ and $\operatorname{Ext}^{r+1}_R(L, N) = 0$ by definition of injective dimension. Hence, $\operatorname{Ext}^r_R(M, N) \neq 0$, and $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > r$. More precisely, $\operatorname{id}_R N = r = \sup\{i \mid \operatorname{Ext}^i_R(M, N) \neq 0\}$. Since $\operatorname{depth}_R M = 0$, this completes the base case.

Inductive Step: Suppose $\operatorname{depth}_R M > 0$, so we can choose some M-regular element $x \in R$, giving an exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

Apply $\operatorname{Hom}_R(-, N)$ to the above:

$$\cdots \longrightarrow \operatorname{Ext}^i_R(M/xM, N) \longrightarrow \operatorname{Ext}^i_R(M, N) \xrightarrow{x} \operatorname{Ext}^i_R(M, N) \longrightarrow$$

$$\operatorname{Ext}^{i+1}_R(M/xM, N) \longrightarrow \cdots$$

By the inductive hypothesis, we have that $\operatorname{id}_R N = \operatorname{depth}_R(M/xM) + \sup\{i \mid \operatorname{Ext}^i_R(M/xM, N) \neq 0\}$. Set $t := \sup\{i \mid \operatorname{Ext}^i_R(M/xM, N) \neq 0\}$, so that for $i \geqslant t$ the above exact sequence implies

$$\cdots \longrightarrow \operatorname{Ext}^t_R(M/xM, N) \longrightarrow \operatorname{Ext}^t_R(M, N) \xrightarrow{x} \operatorname{Ext}^t_R(M, N) \longrightarrow 0$$

is exact, so that $\operatorname{Ext}^t_R(M,N) = x \operatorname{Ext}^i_R(M,N)$. By Nakayama's Lemma, we have that $\operatorname{Ext}^t_R(M,N) = 0$, and likewise $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \geqslant t$. We also have that $\operatorname{Ext}^{t-1}_R(M,N) \neq 0$, since the above shows we have an exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{t-1}_R(M, N) \longrightarrow \operatorname{Ext}^t_R(M/xM, N) \longrightarrow 0$$

so that the vanishing of $\operatorname{Ext}^{t-1}_R(M,N)$ implies the vanishing of $\operatorname{Ext}^t_R(M/xM, N)$, in contradiction with the definition of t.

Thus, we deduce $\sup\{i \mid \operatorname{Ext}_R(M,N) \neq 0\} = t - 1$. Using this and the standard equality $\operatorname{depth}_R M/xM = \operatorname{depth}_R M - 1$, we see:

$$\begin{aligned}
\operatorname{id}_R N &= \operatorname{depth}_R(M/xM) + t \\
&= \operatorname{depth}_R M - 1 + \sup\{i \mid \operatorname{Ext}_R(M,N) \neq 0\} + 1 \\
&= \operatorname{depth}_R M + \sup\{i \mid \operatorname{Ext}_R(M,N) \neq 0\}
\end{aligned}$$

Using that $\operatorname{id}_R N = \operatorname{depth} R$, the above implies

$$\operatorname{depth} R - \operatorname{depth}_R M = \sup\{i \mid \operatorname{Ext}_R(M,N) \neq 0\}$$

which completes the proof.

15. Problem 3.1.25

We have that R is Gorenstein, M is finitely generated.

Assume first that $\operatorname{pd} M = n < \infty$. We can choose a minimal free resolution

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Since R is Gorenstein and each F_i is free, $\operatorname{id} F_i = \operatorname{id} R < \infty$. By the above resolution it is also clear that $\operatorname{id} M \leqslant \operatorname{id} F_i + i < \infty$, and hence $\operatorname{id} M < \infty$, as desired.

Conversely, suppose that $\operatorname{id} M < \infty$. We proceed by induction on the depth of R. When $\operatorname{depth} R = 0$, we have that R is injective ($\operatorname{id} R = \operatorname{depth} R$). Since M is finitely generated with dimension 0, it is finitely presented so that there exists an exact sequence

$$0 \longrightarrow R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0$$

for positive integer m, n. Since R is injective, so is R^n, and hence the above sequence splits so that $R^m = R^n \oplus M$. Then M is the direct summand of a free module, hence projective with $\operatorname{pd} M = 0 < \infty$.

Inductive Step: Suppose $\operatorname{depth} R = n$, and choose some R-regular element x. As M is finitely generated, we have an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

where K denotes the kernel of the surjective map $F \to M$, and F is free.

It suffices to show that $\operatorname{pd} K < \infty$, since any projective resolution for K can be extended to a resolution for M. Since F is free, x is also F regular and hence K-regular. We have the equality:

$$\operatorname{id}_{R/xR} K/xK = \operatorname{id}_R K - 1$$

Since R/xR is also Gorenstein, we can employ the inductive hypothesis to deduce $\operatorname{pd}_{R/xR} K/xK < \infty$. However, $\operatorname{pd}_{R/xR} K/xK = \operatorname{pd}_R K$, hence K has finite projective dimension, so that $\operatorname{pd} M < \infty$ as well, completing the proof.

16. Problem 3.1.26

Suppose that $\operatorname{id} k < \infty$. Then, note:

$$\operatorname{id} k = \sup\{i \mid \operatorname{Ext}^i_R(k,k) \neq 0\} = \operatorname{pd} k$$

Hence $\operatorname{pd} k < \infty$ as well. By the Auslander-Buchsbaum-Serre Theorem, we conclude that R is regular.

17. Problem 3.2.14

(a). Let $E_R(k) := E$ ($E_R(k)$ denotes the injective hull of the residue field k). Note that the natural homomorphism $\varphi : E \to E \otimes_R \widehat{R}$ explicitly takes $x \mapsto x \otimes \widehat{1}$, where $\widehat{1} = \{1 + \mathfrak{m}^t\}_t$.

We first show surjectivity. Let $x \otimes \{a_t + \mathfrak{m}^t\}_t \in E \otimes_R \widehat{R}$. For some positive integer n, we have that $\mathfrak{m}^n x = 0$. We then deduce that $\mathfrak{m}^n \widehat{R}$ annihilates $x \otimes \widehat{1}$, since

$$\mathfrak{m}^n \widehat{R}(x \otimes \widehat{1}) = (\mathfrak{m}^n x) \otimes (\widehat{R}) = 0 \otimes \widehat{R} = 0$$

Consider then $\{a_n + \mathfrak{m}^t\}_t$, where a_n is of the residue class of $\mathfrak{m}^n$ in $\{a_t + \mathfrak{m}^t\}_t$.

$$\text{Claim: } x \otimes \{a_n + \mathfrak{m}^t\}_t = x \otimes \{a_t + \mathfrak{m}^t\}_t$$

The above is equivalent to showing $\{a_n - a_t + \mathfrak{m}^t\}_t(x \otimes \widehat{1}) = 0$. We have two cases: $t > n$ and $t \leqslant n$. If $t \leqslant n$, then by definition of the inverse limit we have that $a_n - a_t + m^t = 0$.

Now, if $t > n$, then $a_n - a_t \in \mathfrak{m}^n$. Since $\mathfrak{m}^n$ annihilates x, the above claim follows.

Finally, using the above, we have:

$$\begin{aligned}\varphi(a_n x) &= (a_n x) \otimes \{1 + \mathfrak{m}^t\}_t \\ &= x \otimes \{a_n + \mathfrak{m}^t\}_t \\ &= x \otimes \{a_t + \mathfrak{m}^t\}_t \quad \text{(by claim)}\end{aligned}$$

So that φ is surjective. It remains to show injectivity. This is much easier, however, as $\widehat{R}$ is faithfully flat over R. Since the identity is clearly injective, we have that $1 \otimes \widehat{1}$ is injective by flatness. Then φ is an isomorphism, so that $E \cong E \otimes_R \widehat{R}$.

(b). Let F denote the injective hull of E over $\widehat{R}$. Then, we want to show that in fact $F = E$, giving that $F = E_{\widehat{R}}(k) = E$.

Note first that F is also the injective hull (over $\widehat{R}$) of the residue field k. We then have that given $x \in F$ there exists a positive integer n such that $\mathfrak{m}^n \widehat{R} x = 0$. Since $k = R/\mathfrak{m} = \widehat{R}/\mathfrak{m}\widehat{R}$, we have the following commutative diagram (rows exact):

$$\begin{tikzcd} 0 \arrow[r] & k = \widehat{k} \arrow[r] \arrow[d] & E = E \otimes_R \widehat{R} \arrow[dl, "\exists \phi"] \\ & E_{\widehat{R}}(\widehat{k}) & \end{tikzcd}$$

Since $F = E_{\widehat{R}}(k)$ is an essential extension, ϕ is a monomorphism, giving us an exact sequence:

$$0 \longrightarrow E \longrightarrow F \longrightarrow F/\operatorname{Ker}\phi \longrightarrow 0$$

By injectivity of E, we have that $F = E \oplus E/\operatorname{Ker}\phi$. However, E and $E/\operatorname{Ker}\phi$ both have an $\widehat{R}$-module structure by part (a), hence we conclude that $E/\operatorname{Ker}\phi = 0$ by indecomposability of F. Thus $E = E_{\widehat{R}}(k)$, as desired.

(c). Note first that since N is finitely generated, $N = N \otimes_R \widehat{R}$. We have the following:

$$\begin{aligned}
\operatorname{Hom}_{\widehat{R}}(\widehat{N}, E) &= \operatorname{Hom}_{\widehat{R}}(N \otimes_R \widehat{R}, E) \\
&= \operatorname{Hom}_R(N, \operatorname{Hom}_{\widehat{R}}(\widehat{R}, E)) \\
&= \operatorname{Hom}_R(N, E)
\end{aligned}$$

Where we've employed Hom-Tensor adjointness and that fact that since $E = E_{\widehat{R}}(k)$ by part (b), Matlis Duality yields $\operatorname{Hom}_{\widehat{R}}(\widehat{R}, E) = T(\widehat{R}) = E$.

18. Problem 3.2.15

Assume that $(R, \mathfrak{m}, k)$ is an Artinian local ring. We proceed with the proof:

$(a) \implies (b)$: Suppose R is Gorenstein. R is Artinian, hence $\operatorname{id} R = \dim R = 0$, so that R is injective. Let M be a finitely generated R-module. M must also be injective, and as R is Artinian, we have an exact sequence

$$0 \longrightarrow R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

Applying $\operatorname{Hom}_R(-, R)$ twice, we note that our sequence remains exact by injectivity and also that R^n and R^m are reflexive since they are free. Thus the resulting exact sequence becomes

$$0 \longrightarrow R^m \longrightarrow R^n \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R) \longrightarrow 0$$

and we immediately see that M must be reflexive.

$(b) \implies (c)$: Suppose that every finitely generated R module is reflexive. Note that $\operatorname{Ann} I = \operatorname{Hom}_R(R/I, R)$, so that $\operatorname{Hom}_R(\operatorname{Ann} I, R) = R/I$ by reflexivity. Since $\operatorname{Ann} R/I = I$, we then see that $I = \operatorname{Ann}(\operatorname{Hom}_R(\operatorname{Ann} I, R))$.

Hence, let $x \in I$. Then $x \operatorname{Hom}_R(\operatorname{Ann} I, R) = 0$. Thus, if we merely choose the inclusion map $i : \operatorname{Ann} I \to R$, the above shows that $xi \equiv 0$ so that $x \in \operatorname{Ann} \operatorname{Ann} I$. Therefore we see $I \subset \operatorname{Ann} \operatorname{Ann} I$.

To show the reverse inclusion, let $x \in \operatorname{Ann} \operatorname{Ann} I$. By definition, $x \operatorname{Ann} I = 0$ giving:

$$\begin{aligned} x \operatorname{Ann} I = 0 &\implies x \operatorname{Hom}(\operatorname{Ann} I, R) = 0 \\ &\implies x \in \operatorname{Ann}(\operatorname{Hom}_R(\operatorname{Ann} I, R) \\ &\implies x \in I \end{aligned}$$

Thus we conclude $\operatorname{Ann} \operatorname{Ann} I = I$.

$(c) \implies (d)$: Assume we are given nonzero ideal I, $J \subset R$. We see:

$$\begin{aligned} I \cap J &= \operatorname{Ann} \operatorname{Ann} I \cap \operatorname{Ann} \operatorname{Ann} J \\ &= \operatorname{Ann}(\operatorname{Ann} I + \operatorname{Ann} J) \end{aligned}$$

Hence $I \cap J = 0$ if and only if $\operatorname{Ann} I + \operatorname{Ann} J = R$. Since I, $J \neq 0$, however, $\operatorname{Ann} I$ and $\operatorname{Ann} J$ are contained in the maximal ideal $\mathfrak{m}$. This immediately gives that $\operatorname{Ann} I + \operatorname{Ann} J \subset \mathfrak{m}$, and hence by the above $I \cap J \neq 0$.

$(d) \implies (a)$: We only need show that R is Cohen-Macaulay and of type $r(R) = 1$. Since R is Artinian, we trivially have that $\dim R = \operatorname{depth} R = 0$, so it remains to prove that $r(R) = 1$.

Recall that type is defined as the vector space dimension of $\operatorname{Ext}_R^{\operatorname{depth} R}(k, R)$. As $\operatorname{depth} R = 0$, we want to find $\dim_k \operatorname{Hom}_R(k, R)$. We recognize $\operatorname{Hom}_R(k, R)$ as the socle $(0 : \mathfrak{m})_R$. Using the fact that R is Artinian gives that $\mathfrak{m}^n = 0$ for some integer n, and hence $(0 : \mathfrak{m})_R$ is nontrivial $\implies \dim_k \operatorname{Hom}_R(k, R) \geqslant 1$. It remains to prove the reverse inequality.

Choose nonzero x, $y \in (0 : \mathfrak{m})_R$. We have that $Rx \neq 0 \neq Ry$, so that by assumption $Rx \cap Ry \neq 0$. Hence we can find elements r, $s \in R$ such that $rx = sy \neq 0$. Note that r, $s \notin \mathfrak{m}$, since else $rx = sy = 0$ by definition of $(0 : \mathfrak{m})_R$. This says that r and s are units and hence $x = r^{-1}sy$.

Taking the above modulo $\mathfrak{m}$, we find that x and y are linearly dependent in the residue field. Since x and y are arbitrary, this says that any two elements are linearly dependent, so that $\dim_k(0 : \mathfrak{m})_R \leqslant 1$. We conclude that $\dim_k(0 : \mathfrak{m})_R = \dim_k \operatorname{Hom}_R(k, R) = 1$ identically, so that R is Cohen-Macaulay of type 1, therefore R is Gorenstein.

19. Problem 3.3.22

This problem is merely definition checking.

(a). This is obviously a group as it behaves the same as a direct sum. For the ring properties,

$$\begin{aligned}((r,m)(s,n))(t,p) &= (rs, sm+rn)(t,p) \\ &= (rst, tsm+trn+rsp) \\ &= (r,m)(st, sp+tn) \\ &= (r,m)((s,n)(t,p))\end{aligned}$$

proves associativity and our identity is obviously $(1,0)$.

(b). Given $r \in R$, send $r \mapsto (r,0)$. This is clearly bijective, so it remains to show it is a homomorphism. Note that $\phi(rs) = (rs,0) = (r,0)(s,0) = \phi(r)\phi(s)$, so $R = R * 0$.

(c). Given $(0, m) \in 0 * M$, we see that $(r, n)(0, m) = (0, rm) \in 0 * M$, so this is an ideal. Given arbitrary $(0, m)$ and $(0, n) \in 0 * M$, we see that $(0, n)(0, m) = (0, 0)$ so that $(0 * M)^2 = 0$.

(d). Suppose we have an ideal $(I, N) \in R * M$, where $I \subset R$ and $N \subset M$. We want to deduce properties of I and N in order for the pair to be an ideal. We see:

$$(R, M)(I, N) = (RI, RN + IM)$$

In order for the above to remain a subset of (I, N), we require that $RI \subset I$, so that I is an ideal, and that $N = M$. Hence, all ideals of $R * M$ are of the form $I * M$ for some ideal $I \subset R$. If R is local, we deduce immediately that $R * M$ is local with maximal ideal $\mathfrak{m} * M$.

(e). By the characterization given in part (d), assume there exists a chain of ideals $I_0 * M \subset I_1 * M \subset \ldots$. This corresponds directly to a chain of ideals $I_0 \subset I_1 \subset \ldots$ in R. If R is Noetherian, this chain stabilizes, and hence $R * M$ is Noetherian.

Choosing a maximal descending chain of prime ideals, this also corresponds to a maximal chain of prime ideals in $R * M$. Immediately we deduce $\dim R = \dim R * M$.

20. Problem 3.3.23

Suppose that $C/\mathbf{x}C$ is a canonical module for $R/\mathbf{x})$, where $\mathbf{x}$ denotes a regular sequence of length n. Since C is assumed maximal Cohen-Macaulay, we only need prove that $\mathrm{id}_R\, C < \infty$ and $r(C) = 1$.

R is Cohen-Macaulay, so that $\dim R = \operatorname{depth} R = \operatorname{id}_R C$. We also have,

$$\begin{aligned} \operatorname{id}_{R/(\mathbf{x})} C/\mathbf{x}C &= \operatorname{depth}_{R/(\mathbf{x})} C/\mathbf{x}C \\ &= \operatorname{depth}_R C - n \\ &= \operatorname{id}_R C - n \end{aligned}$$

Since $\operatorname{id}_{R/(\mathbf{x})} C/\mathbf{x}C < \infty$, we conclude that $\operatorname{id}_R C < \infty$ as well.

Finally, to show $r(C) = 1$, we have the isomorphism (where $d := \operatorname{depth} R$)

$$\operatorname{Ext}_R^d(k, C) \cong \operatorname{Ext}_R^{d-n}(k, C/\mathbf{x}C)$$

We know that $r(C/\mathbf{x}C) = 1$, that is, $\operatorname{rank}_k \operatorname{Ext}_R^{d-n}(k, C/\mathbf{x}C) = 1$, where we've used $\operatorname{depth}_{R/\mathbf{x}} R/\mathbf{x} = \operatorname{depth} R - n$. Then the above isomorphism gives that $r(C) = 1$ as well, completing the proof.

21. Problem 3.3.26

We proceed by induction on the projective dimension of the R-module M. For the base case, set $\operatorname{pd} M = 0$. As R is local, M is free, and also $\operatorname{Tor}_i^R(k, M) = 0$ for all $i > 0$.

Also, R is Gorenstein so that in particular it is Cohen-Macaulay, giving $d := \dim R = \operatorname{depth} R$. Hence, by definition of depth, $\operatorname{Ext}_R^{d-i}(k, M) = 0$ whenever $i > 0$, giving the stated equality when $i > 0$.

Now, when $i = 0$, $\operatorname{Tor}_i^R(k, M) = k \otimes M$. To prove this case, we will show $\operatorname{Tor}_0^R(k, M) = k^{\oplus \mu(M)} = \operatorname{Ext}_R^0(k, M)$. The first equality is trivial, as

$$k \otimes M = k \otimes R^{oplus \mu(M)} = k^{\oplus \mu(M)}$$

For the second equality, we have

$$\begin{aligned}\operatorname{Ext}^d_R(k, M) &= \operatorname{Ext}^d_R(k, R^{\oplus \mu(M)}) \\ &= (\operatorname{Ext}^d_R(k, R))^{\oplus \mu(M)} \\ &= k^{\oplus \mu(M)}\end{aligned}$$

Where the final equality follows from R being Gorenstein. Since $r(R) = 1$, we must have that $\operatorname{Ext}^d_R(k, R) = k$, as they are both vector spaces of dimension 1 over k.

Inductive Step: Suppose now that $\operatorname{pd} M = n$. We chose a free resolution

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

and extract a short exact sequence

$$0 \longrightarrow N \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Applying both $k \otimes -$ and $\operatorname{Hom}_R(k, -)$ to the above, we have the induced sequences

$$\operatorname{Tor}^R_i(k, F_0) \longrightarrow \operatorname{Tor}^R_i(k, M) \longrightarrow \operatorname{Tor}^R_{i+1}(k, M) \longrightarrow \operatorname{Tor}^R_{i+1}(k, F_0)$$

$$\operatorname{Ext}^i_R(k, F_0) \longrightarrow \operatorname{Ext}^i_R(k, M) \longrightarrow \operatorname{Ext}^{i+1}_R(k, M) \longrightarrow \operatorname{Ext}^{i+1}_R(k, F_0)$$

Since F_0 is free, in particular it is flat, so that $\operatorname{Ext}^i_R(k, M) = \operatorname{Tor}^R_i(k, M) = 0$ when $i > 0$. Therefore the above sequences give the isomorphisms

$$\operatorname{Tor}^R_i(k, M) = \operatorname{Tor}^R_{i+1}(k, N)$$

$$\operatorname{Ext}^i_R(k, M) = \operatorname{Ext}^{i+1}_R(k, N)$$

Since $\operatorname{pd} N = n - 1$, we apply the inductive hypothesis to see

$$\operatorname{Tor}^R_i(k, M) = \operatorname{Tor}^R_{i+1}(k, N) = \operatorname{Ext}^{d-i-1}_R(k, N) = \operatorname{Ext}^{d-i}_R(k, M)$$

So that $\operatorname{Tor}^R_i(k, M) = \operatorname{Ext}^{d-i}_R(k, M)$ when $i > 0$. For the case $i = 0$, first note that the exact same reasoning as in the base case gives that

$\operatorname{Ext}_R^d(k, F_0) = k \otimes F_0$, so we have the commutative diagram

$$\begin{tikzcd}
k \otimes N \arrow[r] \arrow[d] & k \otimes F_0 \arrow[r] \arrow[d] & k \otimes M \arrow[d] \\
\operatorname{Ext}_i^d(k, N) \arrow[r] & \operatorname{Ext}_i^d(k, F_0) \arrow[r] & \operatorname{Ext}_i^d(k, M)
\end{tikzcd}$$

The first vertical arrow is an isomorphism by the inductive hypothesis, and the second is an isomorphism by the above. Hence, the third vertical must also be an isomorphism, that is, $\operatorname{Tor}_i^R(k, M) = \operatorname{Ext}_R^{d-i}(k, M)$ when $i = 0$ as well, completing the proof.

22. Problem 3.3.27

By the given assumptions, we see that there exists an epimorphism $\oplus\omega_R \to R$, inducing an exact sequence

$$0 \longrightarrow K \longrightarrow \oplus\omega_R \longrightarrow R \longrightarrow 0$$

As R is free, the above sequence splits, that is, $\oplus\omega_R = R \oplus K$. Since, by definition, ω_R has finite injective dimension, we conclude that $\operatorname{id}_R R < \infty$ as well, so that R is Gorenstein.

23. Problem 3.3.28

(a). We proceed by induction on the dimension of M ($= \dim R$, since M is maximal Cohen-Macaulay). When $\dim M = 0$, M is injective and $\dim R = \dim \omega_R = \operatorname{id} \omega_R = 0$. Thus, ω_R is injective as well, and by 3.3.18 of the book, it is also of rank 1. Therefore $\omega_R = E(R/\mathfrak{m})$, $E(-)$ denoting the injective hull (over R).

However, as M is also injective, we have

$$M = \oplus E(R/\mathfrak{m}) = \oplus\omega_R$$

which completes the base case.

Inductive Step: Set $\dim M = n > 0$. Then, obviously $\operatorname{depth} R > 0$, so we can choose some M-regular element x, inducing a short exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow M/xM \longrightarrow 0$$

As $\operatorname{depth} M/xM = n-1$, we have that $M/xM = \oplus\omega_{R/(x)}$ by the inductive hypothesis. Using 3.3.5 in the book,

$$\omega_{R/(x)} = \omega_R/x\omega_R$$

And, as any regular element of M is also regular in ω_R, we have

$$M/xM = \oplus\omega_R/x\omega_R = (\oplus\omega_R)/x(\oplus\omega_R)$$

and hence we deduce that $M = \oplus\omega_R$, which completes the proof.

(b). Firstly, note that if there exists a finite ω_R resolution of M, then, as each ω_R has finite injective dimension, we deduce that $\operatorname{id} M < \infty$ as well.

To show the converse, we use the hint. Suppose $\operatorname{id} M < \infty$. Note that any maximal Cohen-Macaulay module is isomorphic to a direct sum of canonical modules ω_R. That is, we can find a Cohen-Macaulay approximation:

$$0 \longrightarrow Y_1 \longrightarrow \omega_R^{r_0} \longrightarrow M \longrightarrow 0$$

Then, as Y_1 has finite injective dimension, we can use part (a) to find another Cohen-Macaulay approximation

$$0 \longrightarrow Y_2 \longrightarrow \omega_R^{r_1} \longrightarrow Y_1$$

We continue in this fashion:

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & \\
 & & & Y_1 \arrow[ur] \arrow[dr] & & & \\
\omega_R^{r_2} \arrow[rr] \arrow[dr] & & \omega_R^{r_1} \arrow[rr] \arrow[ur] & & \omega_R^{r_0} \arrow[r] & M \arrow[r] & 0 \\
 & Y_2 \arrow[ur] \arrow[dr] & & & & & \\
0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

We continue this process, adding to the above diagram by taking successive Cohen-Macaulay approximations of each Y_i. Since M has finite injective dimension, this process must terminate past some integer k, giving the desired resolution.

(c).